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PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

Nested Bethe Ansatz for RTT-Algebra of $U_q(\mathrm{sp}(4))$ Type

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Abstract—We study the highest weight representations of the RTT-algebras for the R-matrix of $\mathrm{sp}_q(4)$ type by the nested algebraic Bethe ansatz. It is a generalization of our study for R-matrix of sp(4).

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1. INTRODUCTION

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school [1] provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT-algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors depending on a set of complex variables. The first formulation of the Bethe vectors for the $\mathrm{gl}(n)$-invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [2] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra. Our calculation is some q-generalization of the construction which we published in recent a work [6] for the non-deformed case of sp(4). Our construction of Bethe vectors used the new RTT-algebra $\tilde{\mathcal{A}}_2$ which is defined in Section 2 and is not the RTT-subalgebra of $\mathrm{sp}_q(4)$. This algebra has two RTT-subalgebras of $\mathrm{gl}_q(2)$ type and we study its eigenvectors in Section 3.

Our construction of Bethe vectors is in any sense a generalization of Reshetikhin's results [7]. Another approach to the nested Bethe ansatz for very special representations of RTT-algebras of $\mathrm{sp}(2n)$ type was given by Martin and Ramas [8].

In this note, due to of lack of space, we omit the proofs of the claims. These proofs can be found in our preprint [9].

2. SUMMARY OF THE RESULTS OF THE PAPER [9] FOR THE RTT-ALGEBRA OF $U_q(\mathrm{sp}(4))$ TYPE

Recently, we dealt with the Nested–Bethe ansatz for the RTT-algebra of $U_q(\mathrm{sp}(2n))$ type [9]. In this part, we briefly summarize the results of this work in the case of the RTT-algebra of $U_q(\mathrm{sp}(4))$ type.

The R-matrix of $U_q(\mathrm{sp}(4))$ type has the form

$$\begin{aligned}\mathbf{R}(x) = \frac{1}{\alpha(x)}\Bigg(&\sum_{i,k;i\neq\pm k}\mathbf{E}_i^i\otimes\mathbf{E}_k^k + f(x)\sum_i\mathbf{E}_i^i\otimes\mathbf{E}_i^i\\ &+ f(x^{-1}q^{-3})\sum_i\mathbf{E}_i^i\otimes\mathbf{E}_{-i}^{-i} + g(x)\sum_{k<i}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\\ &- g(x^{-1})\sum_{i<k}\mathbf{E}_k^i\otimes\mathbf{E}_i^k - g(xq^3)\sum_{k<i}q^{k-i}\varepsilon_i\varepsilon_k\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i}\\ &+ g(x^{-1}q^{-3})\sum_{i<k}q^{k-i}\epsilon_i\epsilon_k\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i}\Bigg),\end{aligned} \tag{1}$$

where the census indices take place $i,k = \pm1,\pm2$, $\epsilon_i = \mathrm{sgn}(i)$ and

$$f(x) = \frac{xq - x^{-1}q^{-1}}{x - x^{-1}},\quad g(x) = \frac{x(q-q^{-1})}{x - x^{-1}},$$

$$\alpha(x) = 1 + \frac{q - q^{-1}}{x - x^{-1}}.$$

This R-matrix satisfies the Yang–Baxter equation

$$\mathbf{R}_{1,2}(x)\mathbf{R}_{1,3}(xy)\mathbf{R}_{2,3}(y) = \mathbf{R}_{2,3}(y)\mathbf{R}_{1,3}(xy)\mathbf{R}_{1,2}(x)$$

and is invertible. Therefore, by using the RTT-equation

$$\mathbf{R}_{1,2}(xy^{-1})\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(xy^{-1}),$$

$$\text{where}\quad \mathbf{T}(x) = \sum_{i,k=-n}^{n}\mathbf{E}_i^k\otimes T_k^i(x)$$

we define the RTT-algebra of $U_q(\mathrm{sp}(4))$ type. From the invertibility of the R-matrix we have that the operator

$$H(x) = \mathrm{Tr}(\mathbf{T}(x)) = \sum_{i=-2}^{2} T_i^i(x)$$

fulfills the equations $H(x)H(y) = H(y)H(x)$ for any x and y.

We suppose that in the representation space $\mathcal{W}$ of the RTT-algebra $\mathcal{A}$ there exists a vacuum vector $\omega \in \mathcal{W}$, for which $\mathcal{W} = \mathcal{A}\omega$ and

$$T_k^i(x)\omega = 0 \quad \text{for} \quad i < k,$$
$$T_i^i(x)\omega = \lambda_i(x)\omega \quad \text{for} \quad i = \pm 1, \pm 2.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$, we will look for eigenvectors of $H(x)$.

In [9] we showed that if we restrict our considerations to the space $\mathcal{W}_0 = \mathcal{A}^{(+)}\mathbf{A}^{(-)}\omega \subset \mathcal{W} = \mathcal{A}\omega$, where RTT-subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ are generated by $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i, k = 1, 2$, it is possible to write commutation relations between

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{2} \mathbf{E}_i^k \otimes T_k^i$$

$$\text{and} \quad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{2} \mathbf{E}_{-i}^{-k} \otimes T_{-k}^{-i}(x)$$

in the form of RTT-equations

$$\begin{aligned} &\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1})\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) \\ &= \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1}), \end{aligned} \tag{2}$$

where $\epsilon_1, \epsilon_2 = \pm$ and

$$\mathbf{R}^{(+,+)}(x) = \frac{1}{f(x)}\left(\sum_{i,k=1;i\neq k}^{2} \mathbf{E}_i^i \otimes \mathbf{E}_k^k + f(x)\sum_{i=1}^{2}\mathbf{E}_i^i \otimes \mathbf{E}_i^i + g(x)\mathbf{E}_1^2 \otimes \mathbf{E}_2^1 - g(x^{-1})\mathbf{E}_2^1 \otimes \mathbf{E}_1^2 \right),$$

$$\mathbf{R}^{(-,-)}(x) = \frac{1}{f(x)}\left(\sum_{i,k=1;i\neq k}^{2} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_{-k}^{-k} + f(x)\sum_{i=1}^{2}\mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_{-i}^{-i} + g(x)\mathbf{E}_{-2}^{-1} \otimes \mathbf{E}_{-1}^{-2} - g(x^{-1})\mathbf{E}_{-1}^{-2} \otimes \mathbf{E}_{-2}^{-1} \right),$$

$$\begin{aligned}\mathbf{R}^{(+,-)}(x) = &\sum_{i,k=1;i\neq k}^{2} \mathbf{E}_i^i \otimes \mathbf{E}_{-k}^{-k} + f(x^{-1}q)\sum_{i=1}^{2}\mathbf{E}_i^i \otimes \mathbf{E}_{-i}^{-i} \\ &+ qg(x^{-1}q)\mathbf{E}_2^1 \otimes \mathbf{E}_{-2}^{-1} - q^{-1}g(xq^{-1})\mathbf{E}_1^2 \otimes \mathbf{E}_{-1}^{-2},\end{aligned}$$

$$\begin{aligned}\mathbf{R}^{(-,+)}(x) = &\sum_{i,k=1;i\neq k}^{2} \mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_k^k + f(x^{-1}q^{-3})\sum_{i=1}^{2}\mathbf{E}_{-i}^{-i} \otimes \mathbf{E}_i^i \\ &- q^{-1}g(xq^3)\mathbf{E}_{-2}^{-1} \otimes \mathbf{E}_2^1 + qg(x^{-1}q^{-3})\mathbf{E}_{-1}^{-2} \otimes \mathbf{E}_1^2.\end{aligned}$$

The RTT-equation (2) can be written in the form of a single RTT-equation

$$\tilde{\mathbf{R}}_{1,2}(xy^{-1})\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{T}}_2(y) = \tilde{\mathbf{T}}_2(y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{R}}_{1,2}(xy^{-1}),$$

where

$$\tilde{\mathbf{R}}(x) = \mathbf{R}^{(+,+)}(x) + \mathbf{R}^{(+,-)}(x) + \mathbf{R}^{(-,+)}(x) + \mathbf{R}^{(-,-)}(x),$$
$$\tilde{\mathbf{T}}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x).$$

Since the R-matrix $\tilde{\mathbf{R}}(x)$ satisfies the Yang–Baxter equation

$$\tilde{\mathbf{R}}_{1,2}(x)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{2,3}(y) = \tilde{\mathbf{R}}_{2,3}(y)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{1,2}(x)$$

and is invertible, we can define the RTT-algebra denoted by $\tilde{\mathcal{A}}_2$. If we want to point out that $\mathbf{T}^{(\pm)}(x)$ is an element of the RTT–algebra $\tilde{\mathcal{A}}_2$, we will write

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{2} \mathbf{E}_i^k \otimes \tilde{T}_k^i(x), \ \ \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{2} \mathbf{E}_{-i}^{-k} \otimes \tilde{T}_{-k}^{-i}(x).$$

In the standard way by using (2) we obtain that in the RTT-algebra $\tilde{\mathcal{A}}_2$ the operators $\tilde{H}^{(\pm)}(x)$ and $\tilde{H}^{(\pm)}(y)$,

$$\tilde{H}^{(+)}(x) = \mathrm{Tr}_+(\mathbf{T}^{(+)}(x)) = \sum_{i=1}^{2} \tilde{T}_i^i(x),$$
$$\tilde{H}^{(-)}(x) = \mathrm{Tr}_-(\mathbf{T}^{(-)}(x)) = \sum_{i=1}^{2} \tilde{T}_{-i}^{-i}(x)$$

commute with each other.

We look for Bethe vectors in the form

$$\mathcal{B}(\mathbf{u}) = \langle \mathbf{B}_{1,\ldots,M}(\mathbf{u}), \Phi \rangle,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_M)$ are different complex numbers,

$$\begin{aligned} &\mathbf{B}_{1,\ldots,M}(\mathbf{u})\mathbf{B}_1(u_1) \otimes \mathbf{B}_2(u_2) \otimes \ldots \otimes \mathbf{B}_M(u_M) \\ &= \sum_{i_1,\ldots,i_M,k_1,\ldots,k_M}^{2} \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_M} \otimes \mathbf{f}^{-k_1} \otimes \ldots \otimes \mathbf{f}^{-k_M} \\ &\qquad \otimes T_{-k_1}^{i_1}(u_1)\ldots T_{-k_M}^{i_M}(u_M), \end{aligned}$$

$$\begin{aligned}\Phi = &\sum_{r_1,\ldots,r_M,s_1,\ldots,s_M}^{2} \mathbf{f}^{r_1} \otimes \ldots \otimes \mathbf{f}^{r_M} \otimes \mathbf{e}_{-s_1} \\ &\otimes \ldots \otimes \mathbf{e}_{-s_M} \otimes \Phi_{r_1,\ldots,r_M}^{s_1,\ldots,s_M},\end{aligned}$$

where $\Phi_{i_1,i_2,\ldots,i_M}^{k_1,k_2,\ldots,k_M} \in \mathcal{W}_0$ and $\mathbf{e}_i$ is the basis of space $\mathcal{V}_+$, $\mathbf{e}_{-s}$ is the basis $\mathcal{V}_-$, and $\mathbf{f}^r$ and $\mathbf{f}^{-k}$ are dual bases in spaces $\mathcal{V}_+^*$ and $\mathcal{V}_-^*$, respectively.

In [9] we introduced for any $\mathbf{u}$ the operators

$$\hat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u}) = \hat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu_1^{-1})\dots\hat{\mathbf{R}}^{(+,+)}_{0,M^*}(xu_M^{-1}) \times \mathbf{T}^{(+)}_0(x)\hat{\mathbf{R}}^{(+,-)}_{0,M}(xu_M^{-1})\dots\hat{\mathbf{R}}^{(+,-)}_{0,1}(xu_1^{-1}),$$

$$\hat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u}) = \hat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu_1^{-1})\dots\hat{\mathbf{R}}^{(-,+)}_{0,M^*}(xu_M^{-1})\mathbf{T}^{(-)}_0(x) \times \hat{\mathbf{R}}^{(-,-)}_{0,M}(xu_M^{-1})\dots\hat{\mathbf{R}}^{(-,-)}_{0,1}(xu_1^{-1}),$$

where

$$\hat{\mathbf{R}}^{(+,+)}_{0,1^*}(x) = \frac{1}{f(x^{-1})}\Bigg(\sum_{i,k=1;i\neq k}^{2}\mathbf{E}^i_i\otimes\mathbf{F}^k_k\otimes\mathbf{I}_- + f(x^{-1})\sum_{i=1}^{2}\mathbf{E}^i_i\otimes\mathbf{F}^i_i\otimes\mathbf{I}_- + g(x^{-1})\mathbf{E}^1_2\otimes\mathbf{F}^2_1\otimes\mathbf{I}_- - g(x)\mathbf{E}^2_1\otimes\mathbf{F}^1_2\otimes\mathbf{I}_-\Bigg),$$

$$\hat{\mathbf{R}}^{(-,+)}_{0,1^*}(x) = \sum_{i,k=1;i\neq k}^{2}\mathbf{E}^{-i}_{-i}\otimes\mathbf{F}^k_k\otimes\mathbf{I}_- + f(xq)\sum_{i=1}^{2}\mathbf{E}^{-i}_{-i}\otimes\mathbf{F}^i_i\otimes\mathbf{I}_- + qg(xq) \times \mathbf{E}^{-1}_{-2}\otimes\mathbf{F}^1_2\otimes\mathbf{I}_- - q^{-1}g(x^{-1}q^{-1})\mathbf{E}^{-2}_{-1}\otimes\mathbf{F}^2_1\otimes\mathbf{I}_-,$$

$$\hat{\mathbf{R}}^{(+,-)}_{0,1}(x) = \sum_{i,k=1;i\neq k}^{2}\mathbf{E}^i_i\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-k} + f(x^{-1}q)\sum_{i=1}^{2}\mathbf{E}^i_i\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-i} + qg(x^{-1}q)\mathbf{E}^1_2\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-1}_{-2} - q^{-1}g(xq^{-1})\mathbf{E}^2_1\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-2}_{-1},$$

$$\hat{\mathbf{R}}^{(-,-)}_{0,1}(x) = \frac{1}{f(x)}\Bigg(\sum_{i,k=1;i\neq k}^{2}\mathbf{E}^{-i}_{-i}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-k} + f(x)\sum_{i=1}^{2}\mathbf{E}^{-i}_{-i}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-i} + g(x)\mathbf{E}^{-1}_{-2}\otimes\mathbf{I}^*_+ \otimes\mathbf{E}^{-2}_{-1} - g(x^{-1})\mathbf{E}^{-2}_{-1}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-1}_{-2}\Bigg),$$

and define the operators $\hat{T}^i_k(x;\mathbf{u})$ and $\hat{T}^{-i}_{-k}(x;\mathbf{u})$ by the relationships

$$\hat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u}) = \sum_{i,k=1}^{2}\mathbf{E}^k_i\otimes\hat{T}^i_k(x;\mathbf{u}),$$

$$\hat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u}) = \sum_{i,k=1}^{2}\mathbf{E}^{-k}_{-i}\otimes\hat{T}^{-i}_{-k}(x;\mathbf{u}).$$

For organized M-tuples $\mathbf{u} = (u_1,\dots,u_M)$ denote by $\bar{u}$ the set $\bar{u} = \{u_1,\dots,u_M\}$, define

$$\mathbf{u}_k = (u_1,\dots,u_{k-1},u_{k+1},\dots,u_M),$$

$$\bar{u}_k = \bar{u}\backslash\{u_k\} = \{u_1,\dots,u_{k-1};u_{k+1},\dots,u_M\},$$

$$F(x;\bar{u}^{-1}) = \prod_{k=1}^{M}f(xu_k^{-1}),$$

$$F(x^{-1},\bar{u}) = \prod_{k=1}^{M}f(x^{-1}u_k).$$

One of the main results of [9] is

Proposition 1. Let Φ be a common eigenvector of the operators

$$\hat{H}^{(+)}_{1,\dots,M}(x;\mathbf{u}) = \mathrm{Tr}_0(\hat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u})) = \hat{T}^1_1(x;\mathbf{u}) + \hat{T}^2_2(x;\mathbf{u}),$$
$$\hat{H}^{(-)}_{1,\dots,M}(x;\mathbf{u}) = \mathrm{Tr}_0(\hat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u})) = \hat{T}^{-1}_{-1}(x;\mathbf{u}) + \hat{T}^{-2}_{-2}(x;\mathbf{u})$$

with eigenvalues $\hat{E}^{(+)}_{1,\dots,M}(x;\mathbf{u})$ and $\hat{E}^{(-)}_{1,\dots,M}(x;\mathbf{u})$. If the relations

$$F(u_k^{-1};\bar{u}_k)\hat{E}^{(+)}_{1,\dots,M}(u_k;\mathbf{u}) = F(u_k;\bar{u}_k^{-1})\hat{E}^{(-)}_{1,\dots,M}(u_k;\mathbf{u}) \quad (3)$$

are true for each $u_k\in\bar{u}$, then $\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\Phi\rangle$ is the eigenvector of $H(x) = H^{(+)}(x) + H^{(-)}(x)$ with eigenvalue

$$E_{1,\dots,M}(x;\bar{u}) = F(x^{-1};\bar{u})\hat{E}^{(+)}_{1,\dots,M}(x;\mathbf{u}) + F(x;\hat{u}^{-1})\hat{E}^{(-)}_{1,\dots,M}(x;\mathbf{u}).$$

So to find eigenvectors of the operator $H(x)$, it is enough to find common eigenvectors of the operators $\hat{H}^{(+)}_{1,\dots,M}(x;\mathbf{u})$ and $H^{(-)}_{1,\dots,M}(x;\mathbf{u})$.

Other important results of [9] are the RTT-equations

$$\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})\hat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,M}(x;\mathbf{u})\hat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,M}(y;\mathbf{u}) = \hat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,M}(y;\mathbf{u})\hat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,M}(x;\mathbf{u})\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1}),$$

which hold for any $\epsilon,\epsilon' = \pm$ and for any $\mathbf{u}$. It means that the operators $\hat{T}^i_k(x;\mathbf{u})$ and $\hat{T}^{-i}_{-k}(x;\mathbf{u})$ generate the RTT-algebra $\tilde{\mathcal{A}}_2$ for any $\mathbf{u}$.

Finally, it is shown in [9] that for the vector

$$\hat{\Omega} = \underbrace{\mathbf{f}^1\otimes\dots\otimes\mathbf{f}^1}_{M\times}\otimes\underbrace{\mathbf{e}_{-1}\otimes\dots\otimes\mathbf{e}_{-1}}_{M\times}\otimes\omega$$

we have

$$\hat{T}^1_2(x;\mathbf{u})\hat{\Omega} = 0,\quad \hat{T}^k_k(x;\mathbf{u})\hat{\Omega} = \mu_k(x;\mathbf{u})\hat{\Omega}\ \text{ for } k = 1,2$$
$$\hat{T}^2_{-1}(x;\mathbf{u})\hat{\Omega} = 0,\ \hat{T}^{-k}_{-k}(x;\mathbf{u})\hat{\Omega} = \mu_{-k}(x;\mathbf{u})\hat{\Omega}\ \text{for } k = 1,2,$$

where

$$\mu_1(x;\bar{u}) = \lambda_1(x)F(x^{-1}q;\bar{u}),$$
$$\mu_2(x;\bar{u}) = \lambda_2(x)F(xq^{-1};\bar{u}^{-1}),$$
$$\mu_{-1}(x;\bar{u}) = \lambda_{-1}(x)F(xq;\bar{u}^{-1}),$$
$$\mu_{-2}(x;\bar{u}) = \lambda_{-2}(x)F(x^{-1}q^{-1};\bar{u}),$$

i.e. $\hat{\Omega}$ is a vacuum vector for the representation of the RTT-algebra $\tilde{\mathcal{A}}_2$.

So to find our own vectors of the operator $H(x)$ for the RTT-algebra of $U_q(\mathrm{sp}(4))$ type, just formulate the Bethe ansatz for the RTT-algebra $\tilde{\mathcal{A}}_2$

3. COMMON EIGENVECTORS OF THE OPERATORS $\tilde{H}^{(+)}(x)$ AND $\tilde{H}^{(-)}(x)$ IN THE RTT-ALGEBRA $\tilde{\mathcal{A}}_2$

It is possible from the commutation relations in the RTT-algebra $\tilde{A}_2$ to prove that for each x and y

$$\tilde{T}_1^2(x)\tilde{T}_1^2(y) = \tilde{T}_1^2(y)\tilde{T}_1^2(x),$$
$$\tilde{T}_{-2}^{-1}(x)\tilde{T}_{-2}^{-1}(y) = \tilde{T}_{-2}^{-1}(y)\tilde{T}_{-2}^{-1}(x),$$
$$\tilde{T}_1^2(x)\tilde{T}_{-2}^{-1}(y) = \tilde{T}_{-2}^{-1}(y)\tilde{T}_1^2(x)$$

hold.

Let $\tilde{\omega}$ be a vacuum vector for the representation of the RTT-algebra $\tilde{\mathcal{A}}_2$, i.e. we have

$$\tilde{T}_2^1(x)\tilde{\omega} = \tilde{T}_{-1}^{-2}(x)\tilde{\omega} = 0,$$
$$\tilde{T}_{\pm i}^{(\pm i)}(x)\tilde{\omega} = \mu_{\pm i}(x)\tilde{\omega}, \quad i = 1, 2.$$

Common eigenvectors of the operators $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(x)$ will be searched for in the form

$$|\bar{v};\bar{w}\rangle = \tilde{T}_1^2(v_1)\tilde{T}_1^2(v_2)\ldots\tilde{T}_1^2(v_P)\tilde{T}_{-2}^{-1}(w_1)$$
$$\times\, \tilde{T}_{-2}^{-1}(w_2)\ldots\tilde{T}_{-2}^{-1}(w_Q)\tilde{\omega} = \tilde{T}_1^2(\bar{v})\tilde{T}_{-2}^{-1}(\bar{w})\tilde{\omega},$$

where $\bar{v}$ and $\bar{w}$ are the sets $\bar{v} = \{v_1, v_2, \ldots, v_P\}$ and $\bar{w} = \{w_1, w_2, \ldots, w_Q\}$.

Proposition 2. For any x, $\bar{v}$ and $\bar{w}$ we have

$$\tilde{T}_1^1(x)|\bar{v};\bar{w}\rangle = \mu_1(x)F(x;\bar{v}^{-1})F(xq^{-2};\bar{w}^{-1})|\bar{v};\bar{w}\rangle$$
$$- \sum_{v_r\in\bar{v}}\mu_1(v_r)g(xv_r^{-1})F(v_r;\bar{v}_t^{-1})F(v_rq^{-2};\bar{w}^{-1})|x,\bar{v}_r;\bar{w}\rangle$$
$$+ \sum_{w_s\in\bar{w}}\mu_{-2}(w_s)g(xw_s^{-1}q^{-2})F(w_sq^2;\bar{v}^{-1})F(w_s;\bar{w}_s^{-1})$$
$$\times |x,\bar{v};\bar{w}_s\rangle,$$
$$\tilde{T}_2^2(x)|\bar{v};\bar{w}\rangle = \mu_2(x)F(x^{-1};\bar{v})F(x^{-1}q^2;\bar{w})|\bar{v};\bar{w}\rangle$$
$$+ \sum_{v_r\in\bar{v}}\mu_2(v_r)g(xv_r^{-1})F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w})|x,\bar{v}_r;\bar{w}\rangle$$
$$- \sum_{w_s\in\bar{w}}\mu_{-1}(w_s)g(xw_s^{-1}q^{-2})F(w_s^{-1}q^{-2};\bar{v})$$
$$\times F(w_s^{-1};\bar{w}_s)|x,\bar{v};\bar{w}_s\rangle,$$
$$\tilde{T}_{-1}^{-1}(x)|\bar{v};\bar{w}\rangle = \mu_{-1}(x)F(x^{-1}q^{-2};\bar{v})F(x^{-1};\bar{w})|\bar{v};\bar{w}\rangle$$
$$- \sum_{v_r\in\bar{v}}\mu_2(v_r)g(xv_r^{-1}q^2)F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w})|\bar{v}_r;x,\bar{w}\rangle$$
$$+ \sum_{w_s\in w}\mu_{-1}(w_s)g(xw_s^{-1})F(w_s^{-1}q^{-2};\bar{v})F(w_s^{-1};\bar{w}_s)|\bar{v};x,\bar{w}_s\rangle,$$
$$\tilde{T}_{-2}^{-2}(x)|\bar{v};\bar{w}\rangle = \mu_{-2}(x)F(xq^2;\bar{v}^{-1})F(x;\bar{w}^{-1})|\bar{v};\bar{w}\rangle$$
$$+ \sum_{v_r\in\bar{v}}\mu_1(v_r)g(xv_r^{-1}q^2)F(v_r;\bar{v}_r^{-1})F(v_rq^{-2};\bar{w}^{-1})|\bar{v}_r;x,\bar{w}\rangle$$
$$- \sum_{w_s\in w}\mu_{-2}(w_s)g(xw_s^{-1})F(w_sq^2;\bar{v}^{-1})F(w_s;\bar{w}_s^{-1})|\bar{v};x,\bar{w}_s\rangle.$$

From this statement we obtain for the action of the operators $\tilde{H}^{(\pm)}(x)$

$$\tilde{H}^{(+)}(x)|\bar{v};\bar{w}\rangle = \tilde{T}_1^1(x))|\bar{v};\bar{w}\rangle + \tilde{T}_2^2(x)|\bar{v};\bar{w}\rangle$$
$$= (\mu_1(x)F(x;\bar{v}^{-1})F(xq^{-2};\bar{w}^{-1})$$
$$+ \mu_2(x)F(x^{-1};\bar{v})F(x^{-1}q^2;\bar{w}))|\bar{v};\bar{w}\rangle$$
$$- \sum_{v_r\in\bar{v}}g(xv_r^{-1})(\mu_1(v_r)F(v_r;\bar{v}_r^{-1})F(v_rq^{-2};\bar{w}^{-1})$$
$$- \mu_2(v_r)F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w}))|x,\bar{v}_r;\bar{w}\rangle$$
$$- \sum_{w_s\in\bar{w}}g(xw_s^{-1}q^{-2})(\mu_{-1}(w_s)F(w_s^{-1}q^{-2};v)F(w_s^{-1};\bar{w}_s)$$
$$- \mu_{-2}(w_s)F(w_sq^2;\bar{v}^{-1})F(w_s;\bar{w}_s^{-1}))|x,\bar{v};\bar{w}_s\rangle,$$
$$\tilde{H}^{(-)}(x)|\bar{v};\bar{w}\rangle\tilde{T}_{-1}^{-1}(x)|\bar{v};\bar{w}\rangle + \tilde{T}_{-2}^{-2}(x)|\bar{v};\bar{w}\rangle$$
$$= (\mu_{-1}(x)F(x^{-1}q^{-2};\bar{v})F(x^{-1};\bar{w}) + \mu_{-2}(x)F(xq^2;\bar{v}^{-1})$$
$$\times F(x;\bar{w}^{-1}))|\bar{v};\bar{w}\rangle + \sum_{v_r\in\bar{v}}g(xv_r^{-1}q^2)(\mu_1(v_r)$$
$$\times F(v_r;\bar{v}_r^{-1})F(v_rq^{-2};\bar{w}^{-1})$$
$$- \mu_2(v_r)F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w}))|\bar{v}_r;x,\bar{w}\rangle$$
$$+ \sum_{w_s\in w}g(xw_s^{-1})(\mu_{-1}(w_s)F(w_s^{-1}q^{-2};\bar{v})F(w_s^{-1};\bar{w}_s)$$
$$- \mu_{-2}(w_s)F(w_sq^2;\bar{v}^{-1})F(w_s;\bar{w}_s^{-1}))|\bar{v};x,\bar{w}_s\rangle$$

and the following statement:

Proposition 3. If for each $v_r \in \bar{v}$ and $w_s \in \bar{w}$ the Bethe conditions are fulfilled

$$\begin{aligned}&\mu_1(v_r)F(v_r;\bar{v}_r^{-1})F(v_rq^{-2};\bar{w}^{-1})\\&= \mu_2(v_r)F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w}),\\&\mu_{-1}(w_s)F(w_s^{-1}q^{-2};\bar{v})F(w_s^{-1};\bar{w}_s)\\&= \mu_{-2}(w_s)F(w_sq^2;\bar{v}^{-1})F(w_s;\bar{w}_s^{-1}),\end{aligned} \quad (4)$$

the vectors $|\bar{v},\bar{w}\rangle = \tilde{T}_1^2(\bar{u})\tilde{T}_{-2}^{-1}(\bar{w})\tilde{\omega}$ are common eigenvectors of the operators $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(x)$ with eigenvalues

$$\tilde{E}^{(+)}(x;\bar{v};\bar{w}) = \mu_1(x)F(x;\bar{v}^{-1})F(xq^{-2};\bar{w}^{-1})$$
$$+ \mu_2(x)F(x^{-1};\bar{v})F(x^{-1}q^2;\bar{w}),$$
$$\tilde{E}^{(-)}(x;\bar{v};\bar{w}) = \mu_{-1}(x)F(x^{-1}q^{-2};\bar{v})F(x^{-1};\bar{w})$$
$$+ \mu_{-2}(x)F(xq^2;\bar{v}^{-1})F(x;\bar{w}^{-1}).$$

4. BETHE CONDITIONS AND BETHE EIGENVECTORS FOR THE RTT-ALGEBRA OF $U_q(\mathrm{sp}(4))$ TYPE

In Section 2, we have mentioned that the operators $\hat{T}_k^i(x;\mathbf{u})$ and $\hat{T}_{-k}^{-i}(x;\mathbf{u})$ generate $\mathbf{u}$ the RTT-algebra $\tilde{\mathcal{A}}_2$ for each and the vector $\tilde{\Omega}$ is the vacuum vector with weights

$$\mu_1(x;\bar{u}) = \lambda_1(x)F(x^{-1}q;\bar{u}),$$
$$\mu_2(x;\bar{u}) = \lambda_2(x)F(xq^{-1};\bar{u}^{-1}),$$
$$\mu_{-1}(x;\bar{u}) = \lambda_{-1}(x)F(xq;\bar{u}^{-1}),$$
$$\mu_{-2}(x;\bar{u}) = \lambda_{-2}(x)F(x^{-1}q^{-1};\bar{u}).$$

Proposition 4 says that if for each $v_r \in \bar{v}$ and $w_s \in \bar{w}$ the Bethe conditions are fulfilled

$$\mu_1(v_r;\bar{u})F(v_r;\bar{v}_t^{-1})F(v_rq^{-2};\bar{w}^{-1})$$
$$= \mu_2(v_r;\bar{u})F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w}),$$
$$\mu_{-1}(w_s;\bar{u})F(w_s^{-1};\bar{w}_s)F(w_s^{-1}q^{-2};\bar{v})$$
$$= \mu_{-2}(w_s;\bar{u})F(w_s;\bar{w}_s^{-1})F(w_sq^2;\bar{v}^{-1})$$

the vectors

$$\Phi(\mathbf{u};\bar{v};\bar{w})\hat{T}_1^2(\mathbf{u};\bar{v})\hat{T}_{-2}^{-1}(\mathbf{u};\bar{w})\hat{\Omega}$$

are common eigenvectors of the operators $\hat{H}^{(+)}_{1,\ldots,M}(x;\mathbf{u})$ and $\hat{H}^{(-)}_{1,\ldots,M}(x;\mathbf{u})$ with eigenvalues

$$\hat{E}^{(+)}_{1,\ldots,M}(x;\mathbf{u};\bar{v};\bar{w}) = \mu_1(x;\bar{u})F(x;\bar{v}^{-1})F(xq^{-2};\bar{w}^{-1})$$
$$+ \mu_2(x;\bar{u})F(x^{-1};\bar{v})F(x^{-1}q^2;\bar{w}),$$
$$\hat{E}^{(-)}_{1,\ldots,M}(x;\mathbf{u};\bar{v};\bar{w})\mu_{-1}(x;\bar{u})F(x^{-1}q^{-2};\bar{v})F(x^{-1};\bar{w})$$
$$+ \mu_{-2}(x;\bar{u})F(xq^2;\bar{v}^{-1})F(x;\bar{w}^{-1}).$$

From relation (3) it follows that if for each $u_k \in \bar{u}$ we have

$$F(u_k^{-1};\bar{u}_k)\hat{E}^{(+)}_{1,\ldots,M}(u_k;\mathbf{u};\bar{v};\bar{w})F(u_k;\bar{u}_k^{-1})\hat{E}^{(-)}_{1,\ldots,M}(u_k;\mathbf{u};\bar{v};\bar{w})$$

then the vector

$$\mathfrak{V}(\mathbf{u};\bar{v};\bar{w}) = \langle \mathbf{B}_{1,\ldots,M})(\mathbf{u}),\Phi(\mathbf{u};\bar{v};\bar{w})\rangle \qquad (5)$$

is the eigenvector of the operator $H(x)$. From this we obtain the following theorem:

Theorem. Let the Bethe condition

$$\lambda_1(u_k)F(u_k^{-1};\bar{u}_k)F(u_k^{-1}q;\bar{u}_k)F(u_k;\bar{v}^{-1})F(u_kq^{-2};\bar{w}^{-1})$$
$$= \lambda_{-1}(u_k)F(u_k;\bar{u}_k^{-1})F(u_kq;\bar{u}_k^{-1})F(u_k^{-1}q^{-2};\bar{v})F(u_k^{-1};\bar{w}),$$
$$\lambda_1(v_r)F(v_r^{-1}q;\bar{u})F(v_r;\bar{v}_r^{-1})F(v_rq^{-2};\bar{w}^{-1})$$
$$= \lambda_2(v_r)F(v_rq^{-1};\bar{u}^{-1})F(v_r^{-1};\bar{v}_r)F(v_r^{-1}q^2;\bar{w}),$$
$$\lambda_{-1}(w_s)F(w_sq;\bar{u}^{-1})F(w_s^{-1}q^{-2};\bar{v})F(w_s^{-1};\bar{w}_s)$$
$$= \lambda_{-2}(w_s)F(w_s^{-1}q^{-1};\bar{u})F(w_sq^2;\bar{v}^{-1})F(w_s;\bar{w}_s^{-1})$$

be fulfilled for any $u_k \in \bar{u}$, $v_r \in \bar{v}$ and $w_w \in \bar{w}$, then the vectors (5) are eigenvectors of $H(x)$ with eigenvalues

$$E(x;\bar{u};\bar{v};\bar{w}) = \lambda_1(x)F(x^{-1};\bar{u})F(x^{-1}q;\bar{u})F(x;\bar{v}^{-1})$$
$$\times F(xq^{-2};\bar{w}^{-1}) + \lambda_2(x)F(x^{-1};\bar{u})F(xq^{-1};\bar{u}^{-1})$$
$$\times F(x^{-1};\bar{v})F(x^{-1}q^2;\bar{w}) + \lambda_{-1}(x)F(x;\bar{u}^{-1})$$
$$\times F(xq;\bar{u}^{-1})F(x^{-1}q^{-2};\bar{v})F(x^{-1};\bar{w})$$
$$+ \lambda_{-2}(x)F(x;\bar{u}^{-1})F(x^{-1}q^{-1};\bar{u})F(xq^2;\bar{v}^{-1})F(x;\bar{w}^{-1}).$$

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